MA152 - PRACTICE FINAL

No calculators with a Computer Algebra System (CAS) allowed.

1. Final dy / dx. Apply logarithmic properties if possible.
   Simplify your answer.

a) $y = \ln\left(\dfrac{e^x}{1+e^x}\right) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{1+e^x}$

____

b) $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\ \arcsin x \Rightarrow \quad \dfrac{dy}{dx} = (\arcsin x)^2$

____

c) $\cos x^2 = x e^y \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-2x\sin x^2 - e^y}{xe^y}$

____

d) $y = (c/x)^x \quad \Rightarrow \quad \dfrac{dy}{dx} = \left(\dfrac{c}{x}\right)^x \left[\ln\left(\dfrac{c}{x}\right) - 1\right]$

____

2. Evaluate the derivative of the inverse of f at the given number a

a) $f(x) = \dfrac{1}{4}x^3 + x - 1$ at $a = 3 \Rightarrow (f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(2)} = \dfrac{1}{4}$

b) $f(x) = x^3 - \dfrac{4}{x}$ at $a = 6 \Rightarrow (f^{-1})'(6) = \dfrac{1}{f'(f^{-1}(6))} = \dfrac{1}{f'(2)} = \dfrac{1}{13}$

3. Evaluate the limits

a) $\lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)\tan x = \lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)}{\cot x}\underset{=}{L}\ldots=1$

b) $\lim_{x\to1} x^{1/(1-x)} = \frac{1}{e}$

c) $\lim_{x\to2^+}\left[\frac{8}{x^2-4}-\frac{x}{x-2}\right]= \lim_{x\to2^+}\frac{8-x(x+2)}{x^2-4} = \underset{=}{L}\ldots=-3/2$

4. Find the volume of the solid generated by revolving the region bounded by the graph of $y = x^3 + x + 1,\ y = 1,\ and\ x = 1$ about the line $x = 2$. Sketch the region and the solid.

$$V = 2\pi\int_a^b R(x)h(x)dx = 2\pi\int_0^1 (2-x)(x^3+x)dx = \ldots = \frac{29\pi}{15}$$

5) Find the arc length of the graph of y = ln (cos x) from x = 0 to x = π / 4.

$$L = \int_0^{\pi/4}\sqrt{\sec^2 x}\, dx = \ldots = \ln\sqrt{2}$$

6) Evaluate the integral

a) $\int arc\tan x\, dx = \quad x\, arc\tan x - \int\frac{x}{1+x^2}dx = x\, arc\tan x - \frac{1}{2}\ln(1+x^2)+C$

b) $\int\frac{1}{x\sqrt{4x^2+9}}dx = \int\frac{\frac{3}{2}\sec^2\theta}{\frac{3}{2}\tan\theta\cdot 3\sec\theta}d\theta = \frac{1}{3}\int\csc\theta\, d\theta = \ldots = -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right|+C$

c) $\int\frac{x^2-1}{x^3+x}dx \quad = -\int\frac{1}{x}dx + \int\frac{2x}{x^2+1}dx = \ldots = \ln\left|\frac{x^2+1}{x}\right|+C$

d) $\int\frac{dx}{\sqrt{8+2x-x^2}} \quad = \int\frac{dx}{\sqrt{9-(x-1)^2}} = \sin^{-1}\left(\frac{x-1}{3}\right)+C$

e) $\int\frac{\ln x}{x^2}dx \quad = -\frac{\ln x}{x} + \int\frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

7. Evaluate the integral or show that it is divergent

a) $\int_0^{\infty} \frac{dx}{(x+1)^2(x+2)} = \lim_{b\to\infty} \int_0^b \left(\frac{1}{x+2} - \frac{1}{x+1} + \frac{1}{(1+x)^2} \right) dx = \ldots = 1 - \ln 2$

b) $\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x^4}} dx = \lim_{b\to 0} \int_{-1}^{b} \left(x^{-1/3} + x^{-4/3} \right) dx + \lim_{a\to 0^+} \int_a^1 \left(x^{-1/3} + x^{-4/3} \right) dx$

But $\int_0^1 \frac{x+1}{\sqrt[3]{x^4}} dx = \lim_{a\to 0^+} \int_a^1 \left(x^{-1/3} + x^{-4/3} \right) dx \Rightarrow \infty$ So, given integral diverges

8. Solve the differential equation

a) $y' = \frac{\ln x}{xy + xy^3} \Rightarrow \int (y + y^3) dy = \int \frac{\ln x}{x} dx \Rightarrow y^4 + 2y^2 = 2(\ln x)^2 + C$

or $y^2 = \sqrt{2(\ln x)^2 + C} - 1$

b) $\frac{dx}{dt} = 1 + t - x - tx = (1+t)(1-x) \Rightarrow \int \frac{dx}{1-x} = \int (1+t) dt \Rightarrow x = 1 + Ce^{-(\frac{t^2}{2}+t)}$

9. Determine whether the series converges or diverges. If possible find its sum.

a) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2} \sum \left(\frac{1}{n} - \frac{1}{n+2} \right) = \ldots = \frac{3}{4}$ Converges and its sum is 3/4

b) $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ (Hint: use the integral test) $= \int_0^{\infty} \frac{\ln(x+1)}{x+1} dx = \ldots = \infty$ Diverges

c) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ (Use limit comparison test) $\lim_{n\to\infty} \frac{n^2}{(2n-1)^2} = \frac{1}{4}$ Converges

d) $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$ (Use Ratio Test) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \ldots = \lim_{n\to\infty} \frac{7}{n} = 0$ Converges by Ratio Test

10. Determine whether the series converges conditionally, absolutely or diverges

a) $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ Converges by Alternating Series Test

But $\sum |a_n| = \sum_{n=0}^{\infty} \frac{1}{n+1}$ Diverges by Limit Comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$

Then, given Series Converges Conditionally

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ Given Series Converges by alternating Series Test

But $\sum |a_n| = \sum \frac{1}{\ln(n+1)}$ Diverges by Direct Comparison Test (to Harmonic S)

Therefore, given Series Converges Conditionally

11. a) Determine the number of terms required to approximate the sum with an error of less than 0.0001. b) Find the approximated value of the sum.

i) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $R_n \le \int_N^{\infty} \frac{1}{x^2+1} dx = \ldots = \frac{\pi}{2} - \tan^{-1} N < 10^{-4}$

Take N = 10,000

ii) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ $|R_N| \le a_{n+1} = \frac{1}{(N+1)!} < 0.0001$

Take N = 7

12. Find the Radius and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \quad \text{By ratio test} \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{3}|x-2| < 1 \quad \text{Then}$$

I = [-1, 5) and R = 3

13. A ball is dropped from a height of 6 ft and begins bouncing. The height of each bounce is 3/4 the height of the previous bounce. Find the total vertical distance traveled by the ball.

Distance = 6 + 12 (3/4) $+12(3/4)^2 + 12(3/4)^3 + \ldots = 42\,ft$

14. Find a power series for

a) $f(x) = \cos\sqrt{x} \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \ldots$

b) $f(x) = \sin^2(x)$ (Hint use: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$)

Sin x = $\frac{1}{2} - \frac{1}{2}\cos 2x = \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!} - \frac{2^7}{8!} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} x^{2n}$

15. Using series approximate

a) $\int_0^1 e^{-x^2}\,dx \;=\; \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots\right] dx \ldots = 0.74462$

b) $\int_0^1 \frac{\sin x}{x}\,dx = x - \frac{x^3}{3\times 3!} + \frac{x^5}{5\times 5!} - \frac{x^7}{7\times 7!} + \frac{x^9}{9\times 9!} - \ldots \Big]_0^1$

$= 1 - \frac{1}{3\times 3!} + \frac{1}{5\times 5!} - \frac{1}{7\times 7!} = 0.9461$